C3 January 2008 Solutions by Mr Ely

Jan 08 C3

1. (i) $fg(1)$

$g(1) = 2(1) - 5 = -3$

$f(-3) = (-3)^3 + 4 = -23$

(ii) $x^3 + 4 = 12$

$x^3 = 8$

$x = \sqrt[3]{8} = 2$

2. (i) $x_1 = 3$

$x_2 = \sqrt[3]{31 - \frac{5}{2} \times 3} = 2.8643$ (4dp)

$x_3 = 2.8780$ (4dp)

$x_4 = 2.8767$ (4dp)

$x_5 = 2.8768$ (4dp)

$\therefore \alpha = 2.877$ (3dp)

(ii)

$x = \sqrt[3]{31 - \frac{5}{2}x}$

$x^3 = 31 - \frac{5}{2}x$

$x^3 + \frac{5}{2}x - 31 = 0$

$2x^3 + 5x - 62 = 0$

3. a) $\sec\frac{1}{2}\alpha = 4 \qquad 0 < \alpha < 180$

$\frac{1}{\cos\frac{1}{2}\alpha} = 4$

$\therefore \cos\frac{1}{2}\alpha = \frac{1}{4}$

$\frac{1}{2}\alpha = \cos^{-1}\left(\frac{1}{4}\right) = 75.52^\circ$

$\alpha = 151.0^\circ$ (1dp)

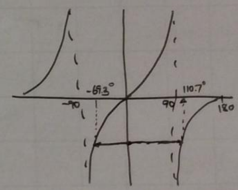
b) $\tan\beta = 7\cot\beta$

$\tan\beta = \frac{7}{\tan\beta}$

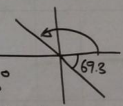
$\tan^2\beta = 7$

$\tan\beta = \pm\sqrt{7}$

$\beta = \tan^{-1}(\sqrt{7}) = 69.3^\circ$ (1dp)



$\beta = \tan^{-1}(-\sqrt{7}) = -69.3^\circ$



Alternative diagram

$= 180 - 69.3^\circ$

$= 110.7^\circ$ (1dp)

$\beta = 69.3^\circ$ or 110.7°

4. i) $V = (h^6 + 16)^{\frac{1}{2}} - 4$

$\frac{dV}{dh} = \frac{1}{2}(h^6 + 16)^{-\frac{1}{2}} \cdot 6h^5$

$= \frac{3h^5}{\sqrt{h^6 + 16}}$

at $h = 2$ $\quad \frac{dV}{dh} = \frac{3(2)^5}{\sqrt{(2)^6 + 16}} = 10.7$ (3sf)

ii) $\frac{dV}{dt} = 8 \qquad \frac{dh}{dt} = ?$

chain rule

$\frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{dh}{dV}$

$\frac{dh}{dt} = 8 \cdot \frac{1}{10.7} = 0.745$ (3sf)

5. a) $\int (3x+7)^9\,dx = \frac{1}{3} \cdot \frac{1}{10}(3x+7)^{10} + c$

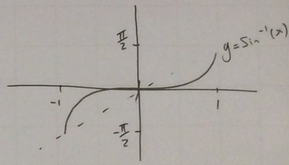
$= \frac{1}{30}(3x+7)^{10} + c$

b) $V = \pi\int y^2\,dx = \pi\int_3^6 \frac{1}{4x}\,dx = \frac{\pi}{4}\int_3^6 \frac{1}{x}\,dx$

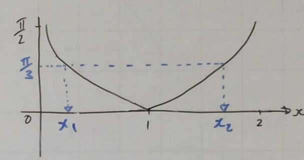
$= \frac{\pi}{4}\left[\ln x\right]_3^6 = \frac{\pi}{4}\left[\ln 6 - \ln 3\right] = \frac{\pi}{4}\ln 2$

6. (i) translation $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

reflection in x axis



(ii)



(iii) $\left|-\sin^{-1}(x-1)\right| = \frac{\pi}{3}$ (using sketch above)

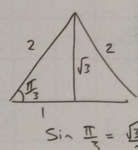
for x_1 $\quad -\sin^{-1}(x-1) = \frac{\pi}{3}$

$\sin^{-1}(x-1) = -\frac{\pi}{3}$

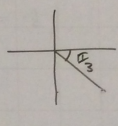
$x - 1 = \sin\left(-\frac{\pi}{3}\right)$

$x - 1 = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$x = 1 - \frac{\sqrt{3}}{2}$



$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$



for x_2 $\quad \sin^{-1}(x-1) = \frac{\pi}{3}$

$x - 1 = \sin\frac{\pi}{3}$

$x = 1 + \frac{\sqrt{3}}{2}$

$\therefore x = 1 \pm \frac{\sqrt{3}}{2}$

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7\. i) if $y = \overset{u}{x}\,\overset{v}{e^{2x}}$

$$\frac{dy}{dx} = e^{2x} + 2e^{2x}x = e^{2x}(1+2x)$$

if $y = \frac{xe^{2x}}{x+k} = \frac{u}{v}$ where $u' = \uparrow$

$$\frac{dy}{dx} = \frac{e^{2x}(1+2x)(x+k) - xe^{2x}}{(x+k)^2}$$

$$= \frac{e^{2x}[(1+2x)(x+k) - x]}{(x+k)^2}$$

$$= \frac{e^{2x}[x+k+2x^2+2kx-x]}{(x+k)^2}$$

$$= \frac{e^{2x}(2x^2+2kx+k)}{(x+k)^2}$$

ii) $\frac{dy}{dx} = 0 \quad \therefore \quad 2x^2+2kx+k = 0$

One solution so $b^2-4ac = 0$ (discriminant)

$$(2k)^2 - 4(2)k = 0$$
$$4k^2 - 8k = 0$$
$$4k(k-2) = 0 \qquad \therefore \cancel{k=0} \text{ or } k=2$$

ii) Continued...

$k=2$

$$2x^2+4x+2 = 0$$
$$x^2+2x+1 = 0$$
$$(x+1)(x+1) = 0$$
$$x = -1$$
$$y = \frac{-e^{-2}}{1} = -e^{-2}$$
$$(-1, -e^{-2})$$

8\. i) $I = \int_0^6 2^x\,dx$

$$= \frac{1}{3}\left[2^0+2^6+4(2^1+2^3+2^5)+2(2^2+2^4)\right]$$
$$= 91$$

ii) $2^x = e^{kx}$ take ln s

$$\ln 2^x = kx$$
$$\cancel{x}\ln 2 = k\cancel{x}$$
$$k = \ln 2 \qquad \therefore 2^x = e^{x\ln 2}$$

$$\int_0^6 2^x\,dx = \int_0^6 e^{x\ln 2}\,dx = \left[\frac{1}{\ln 2}e^{x\ln 2}\right]_0^6 = \frac{1}{\ln 2}\left[e^{6\ln 2}-1\right]$$
$$= \frac{1}{\ln 2}\left[e^{\ln 2^6}-1\right] = \frac{1}{\ln 2}\left[2^6-1\right] = \frac{63}{\ln 2}$$

iii) $\frac{63}{\ln 2} \approx 91$

$$\therefore \ln 2 \approx \frac{63}{91}$$
$$\ln 2 \approx \frac{9}{13}$$

9\. i) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

LHS. $4[\cos\theta\cos 60 - \sin\theta\sin 60][\cos\theta\cos 30 - \sin\theta\sin 30]$

$$4\left[\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right]\left[\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right]$$

$$= 4\left[\frac{\sqrt{3}}{4}\cos^2\theta - \frac{3}{4}\sin\theta\cos\theta - \frac{1}{4}\sin\theta\cos\theta + \frac{\sqrt{3}}{4}\sin^2\theta\right]$$

$$= 4\left[\frac{\sqrt{3}}{4}(\cos^2\theta+\sin^2\theta) - \sin\theta\cos\theta\right] = \sqrt{3} - 4\sin\theta\cos\theta$$

$$= \sqrt{3} - 2\sin 2\theta$$

ii) $4\cos 82.5^\circ\cos 52.5^\circ \quad (\text{let } \theta = 22.5^\circ)$

$$= \sqrt{3} - 2\sin 45 = \sqrt{3} - 2\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-2}{\sqrt{2}} = \frac{\sqrt{6}-2\sqrt{2}}{2}$$

iii) $\sqrt{3} - 2\sin 2\theta = 1$

$$\sin 2\theta = \frac{\sqrt{3}-1}{2}$$

$$2\theta = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) = 21.47, 158.53$$

$$\theta = 10.7^\circ, 79.3 \quad (1dp)$$

iv) $\sqrt{3} - 2\sin 2\theta = k$

$$\sin 2\theta = \frac{\sqrt{3}-k}{2}$$

$$\frac{\sqrt{3}-k}{2} > 1$$
$$-k > 2-\sqrt{3}$$
$$k < \sqrt{3}-2 \quad //$$

$$\frac{\sqrt{3}-k}{2} < -1$$
$$-k < -2-\sqrt{3}$$
$$k > 2+\sqrt{3}$$

$$k < \sqrt{3}-2$$
$$k > 2+\sqrt{3}$$